Equilibrium Pricing in Incomplete Markets - The Multi-Period Model -

STEFAN ANKIRCHNER and ULRICH HORST

Humboldt University Berlin
Department of Mathematics

dbqpl quantitative products laboratory

PIMS, July 2008

Outline

- The multi-period model.
- Reduction to single period models.
- Equilibria in a random walk framework.
- Equilibria in a discretized Brownian motion framework.

The Model

- We consider a dynamic incomplete market model with:
  - a finite set $\mathbb{A}$ of agents endowed with H^a $(a \in \mathbb{A})$
  - a finite sample space $(\Omega, \mathscr{F}, \mathbb{P})$
- At time $t = 1, ..., T$ the agents maximize preference functionals

$$U_t^a : L(\mathscr{F}_T) \to L(\mathscr{F}_t)$$

that is normalized, monotone, translation invariant,

$$U_t^a(X + Z) = U_t^a(X) + Z \quad \text{for all} \quad Z \in L(\mathscr{F}_t),$$

and time consistent, i.e.,

$$U_t^a(X) = U_t^a\left(U_{t+1}^a(X)\right) \quad \text{for all} \quad X \in \mathscr{F}_{t+1}.$$

- The illiquid asset pays a dividend d_t at time $t = 1, ..., T$ so that

$$R_T = \sum_{t=1}^{T} d_T.$$

The illiquid asset will be priced in equilibrium.

Equilibrium in a Dynamic Model

Definition: A partial (in the bond market) equilibrium is a trading strategy $\{(\hat{\eta}_t^a, \hat{\vartheta}_t^a)\}$ along with a price process (R_t) such that:

a) Each agent maximizes her utility from trading:

$$\begin{aligned} & U_t^a\left(H^a + \sum_{t=1}^{T} \hat{\eta}_t^a \Delta S_t + \sum_{t=1}^{T} \hat{\vartheta}_t^a (\Delta R_t + d_t)\right) \\ \geq \quad & U_t^a\left(H^a + \sum_{s=1}^{t} \{\hat{\eta}_s^a \Delta S_s + \hat{\vartheta}_s^a \Delta R_s\} + \sum_{s=t+1}^{T} \{\eta_s^a \Delta S_s + \vartheta_s^a \Delta R_s\}\right) \end{aligned}$$

for all $t = 1, ..., T$ and continuation strategies $\{(\eta_s, \vartheta_s)\}_{s=t+1}^{T}$.

b) The bond markets clears at any point in time:

$$\sum_{a \in \mathbb{A}} \hat{\vartheta}_t^a = 1 \quad \text{for all} \quad t = 1, ..., T.$$

Our goal is to prove the existence of an equilibrium.

Pareto Optimal Risk Allocation

- Assume aggregate utility can be maximized at any time.
- In a one period model this meant:

$$\sum_a U^a\left(H^a + \hat{\eta}^a S_1 + \hat{\vartheta}^a R_1\right) \geq \sum_a U^a\left(\frac{X}{|\mathbb{A}|} + H^a + \eta^a S_1 + \vartheta^a R_1\right)$$

for all strategies that satisfy market clearing because the space

$$\mathbb{S}_1 := \{\eta S_1 + \vartheta R_1 \;:\; \eta, \vartheta \in \mathbb{R}\}$$

across which the agents exchange risks is exogenous as $R_1 = d_1$.

- In a multi-period model risk exchange at time t takes place in

$$\mathbb{S}_t := \{\eta S_1 + \vartheta R_t \;:\; \eta, \vartheta \in \mathbb{R}\}$$

which is generated endogenously because only R_T is initially known!

We need a modification of Condition (A).

Pareto Optimal Risk Allocation

Assumption (A') For all $X^a \in L(\mathscr{F}_T)$ and each subset $E_{t+1} \subset L(\mathscr{F}_{t+1})$ we have

$$\begin{aligned} & \sum_a U_t^a \left(X^a + \hat{\eta}_{t+1}^a \Delta S_{t+1} + \hat{\vartheta}_{t+1}^a R_{t+1} \right) \\ \geq \;& \sum_a U_t^a \left(X^a + \eta_{t+1}^a \Delta S_{t+1} + \vartheta_{t+1}^a R_{t+1} \right) \end{aligned}$$

for all strategies $\{\vartheta_{t+1}^a\} \in L(\mathscr{F}_t)$ that satisfy market clearing.

- By analogy to the static model let $\Phi_t : L(\mathscr{F}_{t+1}) \to L(\mathscr{F}_t)$ be

$$\begin{aligned} \Phi_t(X) \;=\; \sup & \left\{ \sum_a U_t^a \left(\frac{X}{|\mathbb{A}|} + V_{t+1}^a + \eta^a \Delta S_{t+1} + \vartheta^a (R_{t+1} + d_{t+1}) \right) \right. \\ & \left. \text{s.t. } \eta^a, \vartheta^a \in \mathscr{F}_t, \; \sum_a \vartheta^a = 1 \right\} \end{aligned}$$

where V_{t+1}^a denotes the optimal "continuation utility" of $a \in \mathbb{A}$.

THE FUNCTION Φ_t SATISFY SIMILAR REPRESENTATIONS AS Φ.

The representative agent

- By analogy to the static model we have that

$$\Phi_t(X) = \sup_{\xi_t \in \mathscr{D}_{t+1}} \{\mathbb{E}[\xi_t * X] - \varphi_t(\xi_t)\}$$

where $\mathscr{D}_{t+1}$ is the set of all equivalent densities in $L(\mathscr{F}_{t+1})$ and

$$\varphi_t(\xi_t) = \sup_{Y \in L(\mathscr{F}_t)} \{\Phi(Y) - \mathbb{E}[\xi_t * Y | \mathscr{F}_t]\}$$

- In particular, there exists a super-gradient $\hat{\xi}_t$ of Φ_t at zero:

$$\Phi_t(0) = \varphi_t(\hat{\xi}_t)$$

and an equilibrium pricing measure is given defined recursively by

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \hat{\xi}_1 \cdots \hat{\xi}_T.$$

We have a similar characterization as in the static case.

A Characterization and Existence of Equilibrium Result

Theorem: The process $(R_0, ...R_T)$ along with the trading strategy $\{(\eta_t^a, \vartheta_t^a)\}$ is an equilibrium if and only if the following holds:

a) The bond market clears at any time, i.e., $\sum_{a \in \mathbb{A}} \vartheta_t^a = 1$.

b) The representative agent maximizes her utility:

$$\begin{aligned} \Phi_t(0) &= \sum_a U^a(V_{t+1}^a + \eta_{t+1}^a \Delta S_{t+1} + \vartheta_{t+1}^a R_{t+1} + d_{t+1}) \\ &= \varphi_t(\hat{\xi}_{t+1}) - R_t \end{aligned}$$

where the optimal continuation utility is given by

$$V_{t+1}^a = U_{t+1}^a(H^a + \sum_{j=t+2}^{T} \eta_j \Delta S_j + \vartheta_j^a \Delta R_j).$$

c) Asset prices are martingales under the measure $\frac{d\mathbb{P}^*}{d\mathbb{P}} = \hat{\xi}$, i.e.,

$S_t = \mathbb{E}^*[\hat{\xi}_{t+1} * S_{t+1} | \mathscr{F}_t]$ and $R_t = \mathbb{E}[\hat{\xi}_{t+1} * (R_{t+1} + d_{t+1}) | \mathscr{F}_t]$.

UNDER CONDITION (A') AN EQUILIBRIUM EXISTS.

A Characterization and Existence of Equilibrium Result

Corollary: Under Condition (A') an equilibrium exists.

• The equilibrium is defined recursively by backwards induction.

• The problem of dynamic equilibrium pricing can be reduced to a sequence of static models.

• It was key to assume that:
- preferences are translation invariant
- preferences are time consistent.

How can we get more structure into the equilibrium dynamics?

Equilibria in a Random Walk Framework

- Assume $(\mathscr{F}_t)$ is generated by $d \geq 1$ independent random walks:

$$b_t^i = \sum_{s=1}^{t} \Delta b_s^i.$$

- Introducing sufficiently many extra random walks the model is complete and we have the predictable representation property:

$$X = \mathbb{E}[X|\mathscr{F}_t] + \sum_{i=1}^{M} \pi_t^i(X)\Delta b_{t+1}^i \quad (X \in L(\mathscr{F}_{t+1}))$$

where the random coefficients $\pi_t^i(X)$ are given by

$$\pi_t^i(X) = \mathbb{E}[X\Delta b_{t+1}^i|\mathscr{F}_t].$$

Applying this to the functional U_t yield a backward recursion for the preferences.

Equilibria in a Random Walk Framework

- Since the function U_t^a is $\mathscr{F}_t$ translation invariant we see that

$$\begin{aligned} U_t^a(X) &= U_t^a\left(\mathbb{E}[X|\mathscr{F}_t] + \pi_t(X)\cdot\Delta b_{t+1}\right) \\ &= \mathbb{E}[X|\mathscr{F}_t] + f_t^a(\pi_t(X),\omega) \end{aligned}$$

for any $X \in L(\mathscr{F}_t)$ and some adapted convex function $f_t^a(x,\omega)$.

- Applying the predictable representation property to U_{t+1}^a yields:

$$U_{t+1}^a(X) = \mathbb{E}[U_{t+1}^a(X)|\mathscr{F}_t] + Z_{t+1}^a\cdot\Delta b_{t+1}$$

- Since the function U_t^a is time consistent we obtain

$$\begin{aligned} U_t^a(X) &= U_t^a(U_{t+1}^a(X)) \\ &= \mathbb{E}[U_{t+1}^a(X)|\mathscr{F}_t] + f_t^a(Z_{t+1}^a) \end{aligned}$$

Subtracting these equations yields a backward dynamics for the preferences.

Equilibria in a Random Walk Framework

Proposition: The utility functionals satisfy the backward equation:

$$U^a_{t+1}(X) - U^a_t(X) = f^a_t(Z^a_{t+1}) + Z^a_{t+1} \cdot \Delta b_{t+1}, \quad U^a_T(X) = X.$$

- The conditional convolution can be expressed as

$$\Phi_t(X) = \mathbb{E}\left[X + \sum_a V^a_{t+1} + mR_{t+1} | \mathscr{F}_t\right] - f_t(\pi_t(X))$$

where f_t is a point-wise minimizer of $\sum_a f^a_t(x/|\mathbb{A}| + Z^a_t + \dots)$.

Proposition: The conditional super-gradient $\hat{\xi}_{t+1}$ of Φ_t at zero is

$$\hat{\xi}_{t+1} = 1 - \nabla f_t(0) \cdot \Delta b_{t+1}.$$

This yields backward dynamics for the agents' optimal utility.

Equilibria in a Random Walk Framework

- Applying the representation property to $Y \in L(\mathscr{F}_{t+1})$ yields:

$$\mathbb{E}[\hat{\xi}_{t+1} Y | m\mathscr{F}_t] = \mathbb{E}[Y|\mathscr{F}_t] - \pi_t(Y) \cdot \nabla f_t(0).$$

- In particular, we see that

$$\begin{aligned} S_t &= \mathbb{E}[S_{t+1}|\mathscr{F}_t] - Z^S_{t+1} \cdot \nabla f_t(0) \\ R_t &= \mathbb{E}[R_{t+1}|\mathscr{F}_t] - Z^R_{t+1} \cdot \nabla f_t(0). \end{aligned}$$

- Plugging all this into

$$V^a_t = U^a_t \left(V^a_{t+1} + \hat{\eta}^a_{t+1} \Delta S_{t+1} + \hat{\vartheta}^a_{t+1} \Delta R_{t+1} \right), \quad V^a_T = X$$

yields a backward dynamics for prices and optimal utilities.

Equilibrium prices can be computed by means of a stochastic backward difference equation.

Equilibria in a Random Walk Framework

Theorem: An equilibrium price process (R_t) can be computed recursively backwards by:

$$\begin{aligned} R_{t+1} - R_t &= Z^R_{t+1} \cdot \nabla f_t(0) + Z^R_{t+1} \cdot \Delta b_{t+1}, \quad R_T = H \\ V^a_{t+1} - V^a_t &= g^a(Z^R_{t+1}, Z^a_{t+1}) + Z^a_{t+1} \cdot \Delta b_{t+1}, \quad V^a_T = V^a \end{aligned}$$

where

$$Z^R_{t+1} = \pi_t(R_{t+1}) \quad \text{and} \quad Z^a_{t+1} = \pi_t(V^a_{t+1}).$$

Recall that Z^a enters the definition of f_t so the system is coupled.

- It is important that R_t and Z^R_t are not computed simultaneously; in continuous time, they will!

EQUILIBRIUM PRICES CAN BE NUMERICALLY COMPUTED IN AN EFFICIENT WAY.

Equilibria in Brownian Motion Framework

- Assume $(\mathscr{F}_t)$ is generated by independent Brownian motions:

$$\mathscr{F}_t = \sigma(B_s : s = 0, h, ..., i_s * h) \quad t = i_t * h.$$

where $B = (B^1, ..., B^d)$ is a d-dim. standard Brownian motion.

- In this case Ω is no longer finite and we cannot introduce finitely many Brownian motions to complete the market.
- In this case the predictable representation is no longer exact.
- Hence the preference functionals are no longer monotone.
- In this case we can still optimize in every period, but we lose the dynamic programming principle.

THE BEST WE CAN HOPE FOR IS AN APPROXIMATE EQUILIBRIUM.

Equilibria in Brownian Motion Framework

- Define $\{(\hat{\vartheta}^a, \hat{\eta})\}$ and $\{(R_t, V_t)\}_{t\in\mathbb{T}}$ analogously to the random walk framework s.t.:

  a) The market clearing condition holds at any time.

  b) We have one-period optimality:

$$\begin{aligned} & U_t^a\left(V_{t+h}^a + \hat{\vartheta}_{t+h}^a \cdot \Delta S_{t+h} + \hat{\eta}_{t+h}^a \cdot \Delta R_{t+h}\right) \\ \geq \quad & U_t^a\left(V_{t+h}^a + \vartheta_{t+h}^a \cdot \Delta S_{t+h} + \eta_{t+h}^a \cdot \Delta R_{t+h}\right) \,, \end{aligned}$$

  where $V_T^a = H^a$ and

$$\begin{aligned} V_t^a &= U_t^a\left(H^a + \sum_{s>t} \hat{\vartheta}_s^a \Delta S_s + \hat{\eta}_s^a \Delta R_s\right) \\ &= U_t^a\left(V_{t+1}^a + \hat{\vartheta}_{t+h}^a \Delta S_{t+h} + \hat{\eta}_{t+h}^a \Delta R_{t+h}\right) . \end{aligned}$$

We still obtain a "backwards induction dynamics".

Equilibria in Brownian Motion Framework

- The process $\{(R_t, V_t)\}_{t\in\mathbb{T}}$ can again be constructed by backwards induction:

$$\begin{aligned} R_t &= \mathbb{E}[R_{t+h} \mid \mathscr{F}_t] - hZ_t^R \cdot \nabla f_t(0)\,, \qquad & R_T = d \\ V_t^a &= \mathbb{E}[V_{t+h}^a \mid \mathscr{F}_t] - hg_t^a\left(Z_t^R, Z_t^a\right)\,, \qquad & V_T^a = H^a \end{aligned}$$

- This is a descretized version of a coupled system of continuous time BSDEs:

$$dY_t = F(t, Z_t)dt - Z_t dW_t, \quad Y_T = H.$$

- However, we not have a stochastic difference dynamics anymore.

WHAT HAPPENS IF WE LET THE STEP SIZE TEND TO ZERO?